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SYMMETRY AND DEGENERACY

OF

CHARACTERISTIC MODES FOR CONDUCTING BODIES

by

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ABSTRACT:

The notion of symmetry groups is introduced and the representation of such groups is discussed. It is shown that the operator for the eigencurrents on a conducting body is invariant under the group of symmetry operations of the structure. The eigencurrents are shown to provide bases for the irreducible representations of the symmetry group. It is further proven that expansion of the current in terms of functions belonging to the irreducible representations of the symmetry group of the structure leads to block diagonalization of the matrix representation of the operator. Basis functions for bodies of revolution are discussed. Finally, perturbations are considered and it is argued that symmetry determines exactly the splitting of any degenerate resonances of the unperturbed conducting body.

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I. INTRODUCTION

During the past few years, considerable effort has been expended on the development of matrix methods for the solution of electromagnetic field problems [1]. Recently, the theory has been advanced by the introduction of the characteristic mode concept [2]-[4]. Using this approach, the current flow on a body is expressed as a linear combination of characteristic currents or eigencurrents, J_n. Associated with each eigencurrent is a characteristic field or eigenfield, E_n. When the body is excited by a field, the resulting current may be thought of as a linear combination of eigencurrents. Significantly, it has been shown [3] that for electrically small bodies, the true radiated or scattered field is well approximated by only a few characteristic modes. In the case of a cone-sphere radiator, convergence of the gain pattern was achieved with five modes and for a cone-sphere scatterer, convergence of the scattering pattern was achieved with eight modes [3]. This approach is intuitively satisfying and in addition provides considerable insight into the electromagnetic behavior of the body.

Looking to the future, it is clear that this method will be extended to bodies which are more complex and electrically larger. In both cases, in order to achieve the aforementioned convergence, more characteristic modes will be required. These modes will be associated with larger eigenvalues of the operator in the case of electrically large bodies. In order to obtain the necessary accuracy, larger matrices will be required in either case. Computationally, the problem could become staggering in terms of both the speed and storage capabilities of computers. This problem should be viewed with some historical perspective.

Historically, the matrix techniques such as those now being employed in the solution of electromagnetic field problems have been used by physicists for some years. The physicist, of course, is concerned with systems which occur in nature. Nature's systems are highly symmetrical. It was only natural, therefore, that these symmetries were studied for the simplification which they provided in problem solution. It was found that the various symmetry operations of a physical system formed a set which satisfied the mathematical definition of a group. Group theory is a rather well developed branch of algebra [5] and thus the many elegant and powerful mathematical theorems were brought to bear upon the problem. The relationship of group theory to quantum mechanics has been treated by a number of authors [6]-[10].

Diverging for just a moment, we should like to mention that the history of symmetry is a rather interesting subject in itself. The first recorded work in this area dates back to the mid seventeenth century. Early workers were motivated by their observations of naturally occurring crystalline substances and they tried to devise a system of classification according to the symmetry. This work took over two hundred years to complete but by 1890 the 32 point groups and 230 space groups of crystallography had been identified. In the early 1900's it became possible by means of x-ray diffraction to ascertain the symmetry of crystals. Then with the advent of wave mechanics the so called irreducible representations of the space groups became of interest and during the 1930's work commenced on the enumeration of these irreducible representations. An interesting account of the history of symmetry and references to some of the original literature is given by Koster [11].

Now, understanding the current state of theoretical progress with regard to matrix formulation of electromagnetic problems and having discussed briefly the historical development of similar methods in physics it seems quite reasonable that at this point electrical engineers should consider the consequences of symmetry as it relates to their own problems. It is our purpose in the following sections to consider the consequences of symmetry and to show how group theory can be used to simplify matrices when symmetry is present. We shall draw freely upon group theoretical relations but will not obscure the main theme with proofs of theorems etc. as these may be found elsewhere [5]-[10]. We will show that symmetry can provide insight into the nature of the eigencurrents and eigenfields of a conducting body. It also leads to computationally simpler matrix representation of the operator equation.

II. SYMMETRY GROUPS AND THEIR REPRESENTATION

A group is an algebraic structure which may be defined in the following way.

Definition

A non empty set of elements G is said to form a group if in G there is defined a binary operation called the product and denoted by $\cdot$ such that:

(1) $R_1 \cdot R_2$ is in G if R_1 and R_2 are in G (G is closed);

(2) R_1, R_2, R_3 in G implies that $R_1 \cdot (R_2 \cdot R_3) = (R_1 \cdot R_2) \cdot R_3$ ($\cdot$ is associative);

(3) there exists E in G such that $E \cdot R = R \cdot E = R$ for all R in G (E is called the identity element of G);

(4) for every R in G there exists R^{-1} such that $R \cdot R^{-1} = R^{-1} \cdot R = E$ (every element has an inverse).

We shall follow standard practice and discontinue the use of $\cdot$, understanding that in writing R_1R_2 the product is implied.

There are many examples of collections of elements which satisfy the above mathematical definition but we are interested here only in symmetry groups. Given a symmetrical physical object, we identify the various rotations, reflections, etc., which leave the object invariant and the collection of all such symmetry operations forms a group.

In order to understand what is meant by a representation, it is first necessary to introduce the notion of a mapping. The type of mapping in which we are interested establishes a correspondence between one algebraic system and another and preserves the structure.

Definition

A mapping f from a group G onto a group G' is said to be a homomorphism if for all R_1, R_2 in G, $f(R_1 R_2) = f(R_1)\ f(R_2)$.

Now consider the homomorphic mapping of a group G of symmetry elements onto a group G' of n x n matrices (the combining operation for the elements of G'is matrix multiplication).

Definition

Let G be a group, G' a group of n x n matrices, and f a homomorphism from G onto G'. Then G' is said to be a representation of dimension n of the group G.

We will denote the matrix representing an element R of G by D(R). Clearly, the homomorphism f(R) = 1 for all R in G defines a representation in which each element of G is represented by the 1 x 1 identity matrix. This is a simple yet non trivial example since every group has this identity representation.

It is apparent that if we have a representation of a group, another may be obtained from it by means of a similarity transformation. That is, if $D(R_1)\ D(R_2) = D(R_3)$, then it is also true that

$$S\ D(R_1)\ S^{-1} S\ D(R_2)\ S^{-1} = S\ D(R_3)\ S^{-1} \tag{1}$$

where S is a non singular matrix. Representations which are related in this way will not be regarded as distinct.

Definition

Two representations of a group G are said to be equivalent if they are related by a similarity transformation.

If we have two representations of a group G, say $\{D_1\}$ and $\{D_2\}$, it is clear that we may form a third as the direct sum $\{D = D_1 \oplus D_2\}$. In a representation of this type, a group element will be represented by a matrix of the form

$$D(R) = \left(\begin{array}{c|c} D_1(R) & 0 \\ \hline 0 & D_2(R) \end{array}\right) \tag{2}$$

where the matrices of the representations $\{D_1\}$ and $\{D_2\}$ appear along the diagonal.

An equivalent representation may be obtained from a direct sum representation by applying a similarity transformation. The matrices of the representation thus obtained will not, in general, exhibit the block form of (2), and a casual inspection will not reveal that this representation is really the direct sum of two representations of lesser dimension. The notion of decomposing a representations into direct sum form is stated in the following way.

Definition

A representation of a group is reducible if it can be expressed as the direct sum of two or more representations of lesser dimension. A representation is irreducible if it cannot be reduced.

A problem is now apparent. How do we know if a representation may be written as the direct sum of two or more representations of lesser dimension; that is, how do we know if it is reducible? A second problem is the determination of the irreducible representations if the original representation is reducible. These are the two major questions addressed by representation theory. We treat these questions only briefly. To do so we introduce the idea of class conjugacy.

Definition

An element R_1 is said to be conjugate to R_2 with respect to R if $R_1 = RR_2R^{-1}$ (R_1, R_2, R in G).

Lemma: Conjugacy is an equivalence relation.

We do not prove this here but the fact that conjugacy is an equivalence relation means that it will effect a decomposition of a group into distinct subsets.

Definition

The class of elements conjugate to R_1 is that set of elements obtained from RR_1R^{-1} as R runs through the whole of G.

Evidently, we wish to call the subsets we obtain from the partitioning by conjugacy, classes. The reason for this is evident in the next theorem.

Theorem

The number of distinct irreducible representations of a group is equal to the number of classes.

This is a very simple yet powerful result. A finite group has a finite number of classes and the number of its irreducible representations is therefore finite. Let us introduce one more concept.

Definition

Let o(G) be the number of elements in G. Then o(G) is called the order of G.

Theorem

The sum of the squares of the dimensions of the irreducible representations of a group is equal to the order of the group.

If we put this statement in the form of an equation we have

$$\sum_i \ell_i^{\,2} = o(G) \tag{3}$$

where ℓ_i is the dimension of the i^{th} irreducible representation of G. We could not prove either of the above theorems here since many of the details of group structure necessary for their proof have been omitted in an effort to present only the essence of the theory. Together, however, the last two theorems may in some cases be used to determine the dimensions of the irreducible representations.

Let us try to draw some of the preceeding ideas together with an example. Consider the wire square shown in Figure 1. It has the following symmetry operations: rotation by $\pi/2$ (C_4), rotation by π (C_2), rotation by $3\pi/2$ (C_4^{-1}), reflection through the x axis (σ_1), reflection through the y axis (σ_2), reflection through the diagonal with positive slope (σ_4), reflection through the diagonal with negative slope (σ_3) and the identity operation (E) which represents no movement of the sturcture at all. The symbols in parentheses are used to denote the various operations. This group is of order eight and the reader may verify that partitioning by conjugacy produces five distinct classes. There are therefore five irreducible representations whose dimensions when squared must sum to eight. One irreducible representation is the identity representation mentioned earlier. The only possibility here is that there must be four one dimensional and one two dimensional irreducible representations of the group, known as C_{4v} in the literature.

The irreducible representations of C_{4v} are shown in Table I. The symbols on the left which identify the rows of the table denote the irreducible representations. The symbols at the top of the columns identify

TABLE I

Irreducible Representations of C_{4v}

	E	C_2	C_4	C_4^{-1}	σ_1	σ_2	σ_3	σ_4
A_1	1	1	1	1	1	1	1	1
A_2	1	1	1	1	-1	-1	-1	-1
B_1	1	1	-1	-1	1	1	-1	-1
B_2	1	1	-1	-1	-1	-1	1	1
E	$\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$	$\begin{pmatrix}-1 & 0\\0 & -1\end{pmatrix}$	$\begin{pmatrix}0 & -1\\1 & 0\end{pmatrix}$	$\begin{pmatrix}0 & 1\\-1 & 0\end{pmatrix}$	$\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}$	$\begin{pmatrix}-1 & 0\\0 & 1\end{pmatrix}$	$\begin{pmatrix}0 & -1\\-1 & 0\end{pmatrix}$	$\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}$

the various symmetry operations. The entries are the matrices representing group elements in the various irreducible representations. The dual use of E is unfortunate but consistent with the literature.

III. OPERATOR INVARIANCE

Next consider the mathematical formulation of the conducting body problem. An operator equation for the current $\underset{\sim}{J}$ on S is [1],[2]

$$\left[\underset{\sim}{L}(\underset{\sim}{J}) - \underset{\sim}{E}^{i} \right]_{tan} = 0 \tag{4}$$

where the subscript "tan" denotes tangential components on S. The operator $\underset{\sim}{L}$ is defined by

$$\underset{\sim}{L}(\underset{\sim}{J}) = j\omega \underset{\sim}{A}(\underset{\sim}{J}) + \underset{\sim}{\nabla}\Phi(\underset{\sim}{J}) \tag{5}$$

$$\underset{\sim}{A}(\underset{\sim}{J}) = \mu \oint\!\!\oint_{S} \underset{\sim}{J}(\underset{\sim}{r}_s)\psi(\underset{\sim}{r}_f, \underset{\sim}{r}_s)ds_s \tag{6}$$

$$\Phi(\underset{\sim}{J}) = \frac{-1}{j\omega\varepsilon} \oint\!\!\oint_{S} \underset{\sim}{\nabla}_s \cdot \underset{\sim}{J}(\underset{\sim}{r}_s)\psi(\underset{\sim}{r}_f, \underset{\sim}{r}_s)ds_s \tag{7}$$

$$\psi(\underset{\sim}{r}_f, \underset{\sim}{r}_s) = \frac{\exp(-jk\left|\underset{\sim}{r}_f - \underset{\sim}{r}_s\right|)}{4\pi\left|\underset{\sim}{r}_f - \underset{\sim}{r}_s\right|} \tag{8}$$

Now envision the conducting body with current $\underset{\sim}{J}$ and a coordinate system affixed to it. As we perform the various symmetry operations of the structure we move the coordinate system along with the structure but leave the current $\underset{\sim}{J}$ fixed in space. After a symmetry operation the structure looks exactly the same to the current so this is quite permissible. However, the current $\underset{\sim}{J}$ must now be defined in terms of the new coordinate system. For each symmetry operation we define a vector function operator such that

$$V(R)\underset{\sim}{J} = \underset{\sim}{J}' \tag{9}$$

where $\underset{\sim}{J}$ is the current specified in terms of the original coordinates and $\underset{\sim}{J}'$ is the current expressed in terms of the transformed coordinates. The function operators form a group and effect a change of coordinates.

We now consider the action of the V(R) on the operator $\underset{\sim}{L}$. We wish to show that

$$V(R)\ \underset{\sim}{L}(\underset{\sim}{J}) = \underset{\sim}{L}(\underset{\sim}{J}'). \tag{10}$$

First, consider the action of the V(R) on ψ. ψ is a function of $|\underset{\sim}{r}_f - \underset{\sim}{r}_s|$, the scalar distance from field point to source point. Distances are preserved by symmetry operations and thus ψ is invariant under the V(R). That is

$$V(R)\ \psi = \psi \tag{11}$$

Next consider the action of the V(R) on $\underset{\sim}{A}(\underset{\sim}{J})$. The surface S is invariant under symmetry operations and using this along with eq. 11 we have the result

$$V(R)\ \underset{\sim}{A}(\underset{\sim}{J}) = \underset{\sim}{A}(\underset{\sim}{J}'). \tag{12}$$

Further, it is also true that the gradient and divergence operators are invariant under both rotations of frame and change from right to left handedness [6]. Invoking this result along with the invariance of S and (11) it follows that

$$V(R)\ \Phi(\underset{\sim}{J}) = \Phi(\underset{\sim}{J}'). \tag{13}$$

The invariance of $\underset{\sim}{L}$, as expressed in equation (10) is thus established.

IV. CONSEQUENCES OF SYMMETRY

Let us now look at the eigenvalue problem. Following Harrington and Mautz [2], we define

$$Z(\underset{\sim}{J}) = [\underset{\sim}{L}(\underset{\sim}{J})]_{tan} \tag{14}$$

and the eigenvalue problem then takes the form

$$Z(\underset{\sim}{J}_n) = \nu_n \, M(\underset{\sim}{J}_n) \tag{15}$$

where the ν_n are eigenvalues and the $\underset{\sim}{J}_n$ are the eigenfunctions. The weight operator M is chosen equal to R where Z = R + jX to give orthogonality of radiation patterns. With this choice for M, and writing $\nu_n = 1 + j\lambda_n$ eq. (15) becomes

$$X(\underset{\sim}{J}_n) = \lambda_n \, R(\underset{\sim}{J}_n). \tag{16}$$

It is our purpose here to consider the consequences of symmetry with regard to eq. (16).

Suppose we apply the function operators V(R) to both sides of (16). Since we have previously established the invariance of $\underset{\sim}{L}$ and thus Z = R + jX we have

$$X(\underset{\sim}{J}'_n) = \lambda_n \, R(\underset{\sim}{J}'_n) \tag{17}$$

where $\underset{\sim}{J}'_n = V(R)\, \underset{\sim}{J}_n$. This has considerable significance for if $\underset{\sim}{J}_n$ is an eigencurrent belonging to λ_n then so is $V(R)\, \underset{\sim}{J}_n = \underset{\sim}{J}'_n$ for all R in G. Thus, by applying all the V(R) we may generate all the solutions which are degenerate with $\underset{\sim}{J}_n$ by symmetry. Any degenerate functions which cannot be obtained in this way are said to comprise an accidental degeneracy with no origin in symmetry.

If there are n linearly independent eigencurrents $\underset{\sim}{J}_1, \underset{\sim}{J}_2, \ldots \underset{\sim}{J}_n$ with eigenvalue λ then the most general solution with this eigenvalue is a linear combination of these currents. These functions span an n - dimensional subspace in the space of all such currents. Under the V(R), these eigencurrents transform among themselves and the effect of any V(R) is completely determined by specifying its action on each of the $\underset{\sim}{J}_i$ (belonging to λ). Thus,

$$
\begin{aligned}
V(R)\, \underset{\sim}{J}_1 &= a_{11}\, \underset{\sim}{J}_1 + a_{12}\, \underset{\sim}{J}_2 + \ldots + a_{1n}\, \underset{\sim}{J}_n \\
V(R)\, \underset{\sim}{J}_2 &= a_{21}\, \underset{\sim}{J}_1 + a_{22}\, \underset{\sim}{J}_2 + \ldots + a_{2n}\, \underset{\sim}{J}_n \\
&\;\;\vdots \qquad\qquad\qquad\qquad\qquad\qquad \vdots \\
V(R)\, \underset{\sim}{J}_n &= a_{n1}\, \underset{\sim}{J}_1 + a_{n2}\, \underset{\sim}{J}_2 + \ldots + a_{nn}\, \underset{\sim}{J}_n
\end{aligned}
\tag{18}
$$

where the coefficicients a_{ij} may be collected into a square matrix D(R). We see, therefore, that the $\underset{\sim}{J}_i$ belonging to λ form a basis for an n - dimensional representation of the symmetry group of the structure.

We should now ask if this representation $\{D(R)\}$ is reducible. Reducibility depends upon the existence of invariant subspaces. There would be no symmetry based reason for the eigencurrents spanning these different invariant subspaces to have the same eigenvalue so they would not normally exist. If invariant subspaces did exist and their basis currents did have the same eigenvalue, then this would be termed an accidental degeneracy. Accidental degeneracies would be expected to occur very rarely. This result is summarized in the following theorem.

Theorem

If there are no accidental degeneracies, every n - fold degenerate set of eigencurrents of eq. (17) provides a basis for an n - dimensional irreducible representation of the symmetry group of the structure.

V. SIMPLIFICATION OF THE MATRIX OPERATOR

When symmetry exists it may be used to advantage to simplify the matrix representation of the operator Z. Again following Harrington and Mautz [3] we expand the current in a set of functions $\underset{\sim}{W}_j$ as

$$\underset{\sim}{J}_n = \sum_j I_j \underset{\sim}{W}_j \tag{19}$$

where the I_j are coordinate values along the axes labeled by the $\underset{\sim}{W}_j$ in the space which they span. Substituting (19) into (16) we obtain

$$\sum_j I_j X \underset{\sim}{W}_j = \lambda_n \sum_j I_j R \underset{\sim}{W}_j . \tag{20}$$

Taking the inner product of (20) with each of the $\underset{\sim}{W}_i$ we have

$$\sum_j I_j < \underset{\sim}{W}_i , X \underset{\sim}{W}_j > = \lambda_n \sum_j < \underset{\sim}{W}_i , R \underset{\sim}{W}_j >, \; i = 1,2,\ldots \tag{21}$$

where the inner product is given by

$$< \underset{\sim}{A}, \underset{\sim}{B} > = \oiint_S \underset{\sim}{A} \cdot \underset{\sim}{B} \, ds. \tag{22}$$

This is written as a matrix eigenvalue equation

$$[X][I]_n = \lambda_n [R][I]_n \tag{23}$$

where $[I]_n$ is the column matrix of the I_j and

$$R_{ij} = < \underset{\sim}{W}_i , R \underset{\sim}{W}_j > \tag{24}$$

$$X_{ij} = < \underset{\sim}{W}_i , X \underset{\sim}{W}_j >. \tag{25}$$

We will now show that by proper choice and arrangement of the $\underset{\sim}{W}_j$ that [R] and [X] may be block diagonalized.

We introduce without proof the following theorems of group theory.

Theorem

Two functions which belong to different irreducible representations or to different rows of the same unitary representation are orthogonal.

Theorem

Matrix elements of an operator which is invariant under all operations of a group vanish between functions belonging to different irreducible representations or to different rows of the same unitary representation.

Putting this in the form of an equation,

$$< \underset{\sim}{W}_i^{(\alpha,\ell)} , Z \underset{\sim}{W}_j^{(\beta,k)} > = \text{CONST}\ \delta_{\alpha,\beta}\ \delta_{\ell,k} \tag{26}$$

where the superscript (α,ℓ) denotes the ℓ^{th} row of the irreducible representation $\{D_\alpha\}$. This is a result of the greatest significance for it means that by proper choice and ordering of the basis functions, $\underset{\sim}{W}_j$, the matrix representation of the operator can be block diagonalized as previously stated. The resultant form is

$$[Z] = \begin{bmatrix} [Z]^{(\alpha,1)} & & & & \\ & \ddots & & & \\ & & [Z]^{(\alpha,\ell_\alpha)} & & \\ & & & [Z]^{(\beta,1)} & & \\ & & & & \ddots & \\ & & & & & [Z]^{(\beta,\ell_\beta)} \\ & & & & & & \ddots \end{bmatrix} . \tag{27}$$

The dimension of any $[Z]^{(\lambda,k)}$ would be determined by the number of expansion functions chosen to approximate the eigencurrent $\underset{\sim}{J}^{(\lambda,k)}$ belonging to the k^{th} row of $\{D_\lambda\}$. Computationally, of course, the eigencurrents of symmetry species

(λ,k) may be solved for individually and the corresponding eigenvalues determined. For highly symmetric bodies the reduction in computational effort is evident.

VI. EXPANSION OF CURRENT ON BODIES OF REVOLUTION WITH SYMMETRY $C_{\infty v}$

We will now illustrate some of the previous theory for bodies of revolution. Our choice is motivated by the treatment of the cone-sphere by Harrington and Mautz [3].

The symmetry group of a body of revolution is the axial rotation group $C_{\infty v}$. Its operations consist of the set of all rotations about the axis of symmetry (here assumed to be the z-axis) and the set of all reflections through planes containing the axis of symmetry. The group is infinite and has an infinite but denumerable set of irreducible representations. We introduce the following notation:

C_φ : rotation by φ (28a)

σ_x : reflection through x-z plane (28b)

σ_φ : reflection through plane containing z axis and making an angle φ with the x axis. (28c)

These operations completely characterize the group.

We now note that

$$\sigma_\varphi = \sigma_x C_{2\varphi} . \tag{29}$$

Other relations are more obvious. For example,

$$C_{\varphi_1} C_{\varphi_2} = C_{\varphi_1 + \varphi_2} \tag{30a}$$

$$\sigma_\varphi^{\ 2} = \sigma_x^{\ 2} = E \tag{30b}$$

and so forth.

There are an infinite number of irreducible representations and we cannot list them but we can clearly establish their possible forms by writing down the form of the matrices representing the "typical" elements (28). These are listed in Table II. From the table it is evident that

TABLE II. IRREDUCIBLE PREPRESENTATONS OF $C_{\infty v}$

	C_φ	σ_x
A_1	1	1
A_2	1	-1
E_1	$\begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
E_2	$\begin{pmatrix} \cos 2\varphi & -\sin 2\varphi \\ \sin 2\varphi & \cos 2\varphi \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
$\vdots$		
E_n	$\begin{pmatrix} \cos n\varphi & -\sin n\varphi \\ \sin n\varphi & \cos n\varphi \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

there are two one dimensional irreducible representations, A_1 and A_2. These are distinguished by the fact that functions belonging to A_1, the identity representation, are invariant under the reflection σ_x. Equation (29) establishes the transformation properties of such functions under the σ_φ. The remaining representations are two dimensional E types and are distinguished by the integer n. Functions belonging to the first row of any $D_{E_n}(R)$ must be invariant under reflection σ_x while those belonging to the second row go into the negative of themselves.

For the cone-sphere problem, Harrington and Mautz chose the following sets of expansion functions [3]:

$$\{\underset{\sim}{u}_t\ f_i(t),\ \underset{\sim}{u}_t\ f_i(t)\cos n\varphi,\ \underset{\sim}{u}_\varphi\ f_i(t)\sin n\varphi\} \tag{31}$$

$$\{\underset{\sim}{u}_\varphi\ f_i(t),\ \underset{\sim}{u}_t\ f_i(t)\sin n\varphi,\ -\underset{\sim}{u}_\varphi\ f_i(t)\cos n\varphi\} \tag{32}$$

where $\underset{\sim}{u}_\varphi$ is a unit vector in the φ direction and $\underset{\sim}{u}_t$ is a unit vector which is everywhere perpendicular to $\underset{\sim}{u}_\varphi$ on S and directed along the contour, C, which generates the body. The variable t denotes distance along this same contour. Harrington and Mautz note that (32) and (33) are orthogonal sets. We will now show that this is so because the expansion functions transform according to the distinct irreducible representations of the symmetry group $C_{\infty v}$. They are therefore symmetrized basis functions and a proper choice for any body of revolution with no higher symmetry.

For economy we define the following quantities:

$$\underset{\sim}{W}_i^{(A_1)} = \underset{\sim}{u}_t\ f_i(t) \tag{33a}$$

$$\underset{\sim}{W}_i^{(A_2)} = \underset{\sim}{u}_\varphi\ f_i(t) \tag{33b}$$

$$\underset{\sim}{W}_i^{(E_n,1)} = \underset{\sim}{u}_t\ f_i(t)\cos n\varphi + \underset{\sim}{u}_\varphi\ f_i(t)\sin n\varphi \tag{33c}$$

$$\underset{\sim}{W}_i^{(E_n,2)} = \underset{\sim}{u}_t\ f_i(t)\sin n\varphi - \underset{\sim}{u}_\varphi\ f_i(t)\cos n\varphi \tag{33d}$$

First, consider the functions $\underset{\sim}{W}_i^{(A_1)}$. These are invariant under both rotations and reflections and belong to A_1. The functions $\underset{\sim}{W}_i^{(A_2)}$ are invariant under rotations but change sign under reflection because the sense of the unit vector $\underset{\sim}{u}_\varphi$ is reversed. These functions belong to A_2. The remaining functions $\underset{\sim}{W}_i^{(E_n,1)}$ and $\underset{\sim}{W}_i^{(E_n,2)}$ belong to the first and second rows of the E_n. For example, under rotation by φ' we have

$$\begin{bmatrix} \cos n\varphi' & -\sin n\varphi' \\ \sin n\varphi' & \cos n\varphi' \end{bmatrix} \begin{bmatrix} \underset{\sim}{W}_i^{(E_n,1)}(\varphi) \\ \underset{\sim}{W}_i^{(E_n,2)}(\varphi) \end{bmatrix} = \begin{bmatrix} \underset{\sim}{W}_i^{(E_n,1)}(\varphi+\varphi') \\ \underset{\sim}{W}_i^{(E_n,2)}(\varphi+\varphi') \end{bmatrix} \tag{34}$$

and upon reflection, σ_x,

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \underset{\sim}{W}_i^{(E_n,1)} \\ \underset{\sim}{W}_i^{(E_n,2)} \end{bmatrix} = \begin{bmatrix} \underset{\sim}{W}_i^{(E_n,1)} \\ -\underset{\sim}{W}_i^{(E_n,2)} \end{bmatrix} . \tag{35}$$

The expansion of the currents using these basis functions leads to symmetrized currents,

$$\underset{\sim}{J}^{(A_1)} = \sum_i I_i \underset{\sim}{W}_i^{(A_1)} \tag{36a}$$

$$\underset{\sim}{J}^{(A_2)} = \sum_i I_i \underset{\sim}{W}_i^{(A_2)} \tag{36b}$$

$$\underset{\sim}{J}^{(E_n,1)} = \sum_i I_i \underset{\sim}{W}_i^{(E_n,1)} \tag{36c}$$

$$\underset{\sim}{J}^{(E_n,2)} = \sum_i I_i \underset{\sim}{W}_i^{(E_n,2)} \tag{36d}$$

which also provide bases for the irreducible representations of the symmetry group $C_{\infty v}$. Proper ordering of these expansion functions (33) therefore leads to block diagonalization of the matrix representation of the operator Z. For this reason the eigencurrents belonging to the various irreducible representations may be solved for separately using the expansions (36). Classification of eigencurrents by symmetry species is obviously an extremely valuable tool both conceptually and computationally.

VII. PERTURBATIONS

If a highly symmetrical body is perturbed in some way, the symmetry may be lowered. In such a case, if the perturbation is small the eigencurrents of the unperturbed body will remain approximately correct eigencurrents of the perturbed body. The eigenvalues will be shifted from their original values, however. The new eigenvalues may be computed using perturbation theory, of course, but the exact degree of degeneracy remaining among any of the n-fold degenerate sets of eigencurrents of the unperturbed body is determined with ultimate precision by symmetry.

We have seen in the preceeding sections how each n-fold degenerate set of eigencurrents of the unperturbed body provides a basis for an n - dimensional irreducible representation of the symmetry group of the structure. These currents must also provide bases for representations of the symmetry group of the perturbed structure since this will be a subgroup of the symmetry group of the unperturbed body. Representations so formed will in general be reducible, however. It is by decomposition of these representations into irreducible representations that we may determine the degree of residual degeneracy.

For example, suppose we placed a small bump on the cone-sphere thereby making it completely asymmetrical. In this case the eigencurrents would approximate those of the rotationally symmetric body as found from (36) but the degeneracy of eigencurrents belonging to the two rows of the representaions E_n would in all cases be lifted. This means that each doubly degenerate resonance (resonance occurs for those frequencies for which $\lambda = 0$) of the cone-sphere would be split into two closely spaced resonances. If we were to consider scattering from the perturbed cone-sphere we would find that strong scattering would now occur at two closely spaced but distinct frequencies in the vicinity of those frequencies where previously a single peak in the cross-section was observed.

Other more striking examples of splitting of structure resonances could be cited. This subject cannot be treated here without making the presentation considerably longer, however. We will thus terminate the discussion with the thought that this could have very interesting applications insofar as target identification through cross-section measurements is concerned.

VIII. CONCLUSIONS

It has been shown that any symmetry possessed by a conducting body must be reflected in the eigencurrents of the structure. It has also been demonstrated that the group theoretical machinery provides a means of simplifying the form of the matrix representation of the operator if the body has symmetry. This simplification takes the form of block diagonalization of the matrix equation. The submatrices may be solved individually and it therefore becomes possible to treat electrically large and geometrically complex bodies. Failure to capitalize upon symmetries would lead to a matrix of computationally unmanageable size.

We have drawn liberally from the conducting body work of Harrington and Mautz since this provided a convienent means of developing the presentation. It should be evident, however, that the concepts presented here are applicable to a wide variety of problems and not just the conducting body problem.

Finally, we have presented only the essential details of group theory. The subject must be studied in some depth if it is to be fully understood and applied. We hope we have motivated some readers in this direction.

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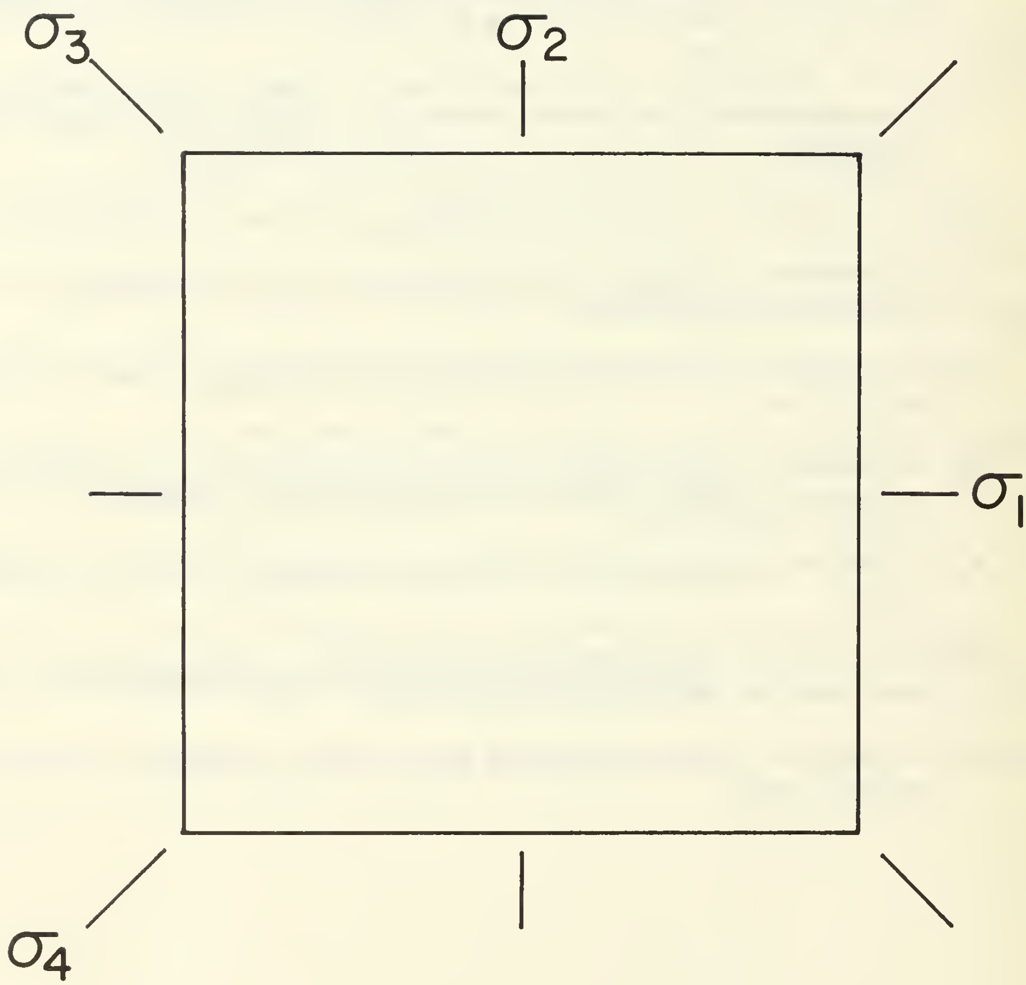
FIGURE 1.



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ABSTRACT

The notion of symmetry groups is introduced and the representation of such groups is discussed. It is shown that the operator for the eigencurrents on a conducting body is invariant under the group of symmetry operations of the structure. The eigencurrents are shown to provide bases for the irreducible representations of the symmetry group. It is further proven that expansion of the current in terms of functions belonging to the irreducible representations of the symmetry group of the structure leads to block diagonalization of the matrix representations of the operator. Basis functions for bodies of revolution are discussed. Finally, perturbations are considered and it is argued that symmetry determines exactly the splitting of any degenerate resonances of the unperturbed conducting body.

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